CENTRE DE RECHERCHE SUR LES TRANSPORTS
(C.R.T.)
CENTRE FOR RESEARCH ON TRANSPORTATION

UNIVERSITÉ DE MONTRÉAL
ÉCOLE DES HAUTES ÉTUDES COMMERCIALES
ÉCOLE POLYTECHNIQUE

CENTRE DE RECHERCHE SUR LES TRANSPORTS
UNIVERSITÉ DE MONTRÉAL
C.P. 6128, SUCCURSALE A
MONTRÉAL
CANADA H3C 3J7

TÉLÉPHONE: (514) 343-7575
TÉLEX: (21) 05-561325
EMAIL: crt@crt.umontreal.ca
TÉLÉCOPIEUR (FAX): (514) 343-7121

CRT-739

Sensitivity Analyses of Minimax Flow Centers in Trees

by
H.A. Eiselt[1]
M. Gendreau[2]

[1] University of New Brunswick, Faculty of Administration, P.O. Box 4400, Fredericton, N.B., Canada E3B 5A3
[2] Centre de recherche sur les transports, Université de Montréal, C.P. 6128, succursale A, Montréal, Québec, Canada H3C 3J7

Centre de recherche sur les transports — Publication #739

December 1990
Revised September 1991

2-00708/

Dépôt légal – 4ème trimestre 1991

ABSTRACT

A minimax flow center in a graph G is a point on G with the following property. Let all nonnegative supplies given at the vertices of G be shipped to some fixed point on G according to a flow pattern that minimizes the largest flow on any of the given edges. If z denotes the value of this largest edge flow, then the minimax flow center of G is that point which minimizes z. In this paper we examine various aspects of the sensitivity of minimax flow centers in trees. In particular, we investigate the optimal addition of an edge, the optimal location of a second facility as well as the optimal relocation of a facility given that an edge is introduced.

Keywords: Minimax flow center, location problem, flow problem, sensitivity analysis.

RÉSUMÉ

Étant donné un graphe, des offres d'un produit aux sommets de ce graphe et un point donné du graphe, on envoie toutes les quantités du produit présentes dans le graphe à ce point de façon à minimiser le plus grand flot résultant sur les arêtes. Soit z la valeur de ce flot, on appelle centre de flot minimax du graphe, le point donné pour lequel z est minimal. Dans cet article, on étudie diverses questions relatives à la sensibilité des centres de flot minimax. On considère en particulier l'ajout optimal d'une arête, la localisation optimale d'un second centre, ainsi que la relocalisation optimale du centre après ajout d'une arête.

Mots-clés: centre de flot minimax, problèmes de localisation, problèmes de flot, analyse de sensibilité.

1. Introduction

In this paper the following problem is studied. Let $T = (V, E, s)$ be an undirected tree where $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices, $E = \{(v_i, v_j) : v_i, v_j \in V\}$ is the set of edges and $s(v_j) \in R_+, j = 1, ..., n$, are supplies assigned to the vertices. It is now the goal to determine the minimax flow center or MMFC of the tree which is defined as a point of T (i.e. a vertex or a point on an edge) with the following property. If all supplies are shipped on their (unique) paths to the MMFC, then the largest flow on any one edge of the tree is as small as possible.

Defining $f_p(v_i, v_j)$ as the flow on edge $(v_i, v_j) \in E$ from v_i to v_j, if the facility is located at v_p, the problem of finding a minimax flow center can be formally stated as follows. Find v_q and $f_q(v_i, v_j), f_q(v_j, v_i), \forall (v_i, v_j) \in E$, which solve

$$\underset{p}{\text{minimize}} \; z = \max_{(v_i, v_j) \in E} \{\max \{(f_p(v_i, v_j), f_p(v_j, v_i))\}\} \tag{1}$$

$$s.t. \sum_{j:(v_i, v_j) \in E} [f_p(v_i, v_j) - f_p(v_j, v_i)] = \begin{cases} -\sum_{j \neq p} s(v_j), & i = p \\ s(v_i), & i \neq p \end{cases} \tag{2}$$

$$f_p(v_i, v_j), f_p(v_j, v_i) \geq 0, \; (v_i, v_j) \in E. \tag{3}$$

In this above formulation, it should be noted that for every edge (v_i, v_j) at most one of the two directed flows $f_p(v_i, v_j)$ and $f_p(v_j, v_i)$ is positive in any optimal solution. This implies that the quantity $\max \{f_p(v_i, v_j), f_p(v_j, v_i)\}$ is indeed the flow on the undirected edge (v_i, v_j) and, therefore, the objective (1) corresponds to the minimization of the largest undirected flow in E. In the following, we will use the

notation $h_p(v_i, v_j)(\triangleq \max\{f_p(v_i, v_j), f_p(v_j, v_i)\})$ for the flow on the undirected edge (v_i, v_j). The objective (1) can then be rewritten as

$$\text{minimize} \quad z = \max_{(v_i, v_j) \in E} \{h_p(v_i, v_j)\} \tag{1'}$$

This problem is considerably easier to solve in trees than in general graphs. The reason is that in the latter case a minimax flow pattern has to be determined for each point under consideration for the MMFC. This is unnecessary in trees since for a given vertex the flow pattern of all supplies to that vertex is unique.

Flow and location problems with minimax objective are well known and have extensively been discussed in the literature. But unlike the bottleneck and center problems as described by Gondran and Minoux (1984) or Handler and Mirchandani (1979) where an attribute associated with the most distant point or the longest path of some flow pattern is minimized, we minimize the magnitude of the largest edge flow itself. This type of objective in the context of network flows has been studied by Minoux (1976), Ichimori et al. (1981) and in the context of transportation problems by Ahuja (1986). Solution methods for the corresponding integer versions were described independently by Finke and Röck (1983) and Tamir (1982). More recently, the equivalence of a sharing problem and weighted minimax flow problems has been shown by Fujishige et al. (1986). For minimax location problems, the reader is referred to Eiselt (1985) and Eiselt and Gendreau (1990, 1991).

Similar to Hakimi's (1964) theorem on medians, Eiselt and Pederzoli (1984) have shown for general graphs that a minimax flow center is not necessarily unique but that at least one MMFC is located at a vertex. This implies the existence of polynomial

algorithms for the minimax flow center problem. Selecting one vertex as tentative MMFC, a minimax flow pattern is determined in $O(n^6)$ by the method of Ichimori et al. (1981). This is then repeated for all vertices and the best solution is chosen, resulting in an overall complexity of $O(n^7)$ for this brute-force approach. Clearly, finding an MMFC in a tree does not require solving a minimax flow problem, but the search for minimax flow centers in trees can be accelerated by some considerations whose basis is derived in the next section.

This paper is organized as follows. In the second section we restate and derive some useful results. The third section discusses the optimal location of an additional facility. The fourth section is devoted to the optimal addition of an edge.

2. Some Basic Results

In this section we review some fundamental results of MMFC's and a number of extensions. The first result concerns the determination of minimax flow centers in trees. Let $v_q \in V$ be any vertex in the tree and denote its degree by $d(v_q)$. Then the removal of v_q and all edges adjacent to v_q decomposes the tree into a forest with subtrees $T_1^q, T_2^q, ..., T_{d(v_q)}^q$. Let the subtrees T_k^q be ordered with respect to nonincreasing supplies, i.e. $s(T_1^q) \geqq s(T_2^q) \geqq ... \geqq s\left(T_{d(v_q)}^q\right)$, where $s(T_k^q) = \sum_{v_j \in T_k^q} s(v_j)$. The supply of the entire tree $s(T)$ is defined accordingly. Then it can be shown that

Theorem 1 : The vertex v_q is a minimax flow center if and only if

$$s(T_1^q) \leqq 1/2\, s(T) \tag{4}$$

This implies that in trees MMFC's and medians coincide and that MMFC's can thus be determined using Goldman's (1971) algorithm (see Eiselt and Gendreau, 1990, for further details).

The sensitivity of the previous result is easily evaluated. The vertex v_q will remain an MMFC as long as condition (4) is satisfied even if the order of the subtrees changes. Consider any vertex $v_i \in T_k^q$, for some $k \in \{1, ..., d(v_q)\}$. If $s(v_i)$ increases by $\epsilon > 0$, then so do $s(T_k^q)$ as well as $s(T)$. Then v_q remains the MMFC as long as $s(T_k^q) + \epsilon \leqq 1/2[s(T) + \epsilon]$ for all k, or

$$\epsilon \leqq s(T) - 2s(T_k^q),\ k = 1, ..., d(v_q). \tag{5}$$

Consequently, the solution is most sensitive with respect to changes of supplies of vertices in T_1^q. Similarly, if $s(v_i), v_i \in T_k^q$ for some k, were to decrease by $\epsilon > 0$, then the right-hand side of relation (4) decreases and the conditions are

$$\begin{aligned} &\epsilon \leqq \min\left\{s(v_i); s(T) - 2s(T_2^q)\right\} \quad \text{if } k = 1 \\ \text{and} \quad &\epsilon \leqq \min\left\{s(v_i); s(T) - 2s(T_1^q)\right\} \quad \text{if } k \neq 1 \end{aligned} \tag{6}$$

It will be useful to prove that

Lemma 2 : If v_q is a minimax flow center, then for every edge $(v_i, v_j) \in E$,

$h_q(v_i, v_j) \leqq h_p(v_i, v_j), \forall v_p \in V$.

Proof : For an arbitrary edge $(v_i, v_j) \in E$, let $T_i^{ij}(T_j^{ij})$ be the subtree of T containing v_i (v_j) when edge (v_i, v_j) is removed from T. Without loss of generality, suppose

that $v_q \in T_j^{ij}$, so that $f_q(v_i, v_j) = s(T_i^{ij})$. Note that $f_p(v_i, v_j) = f_q(v_i, v_j), \forall v_p \in T_j^{ij}$. Consider now $v_p \in T_i^{ij}$. In this case, $f_p(v_j, v_i) = s(T_j^{ij}) = s(T) - s(T_i^{ij})$. Since $T_i^{ij} \subset T_k^q$ for some k and $s(T_k^q) \leqq 1/2\ s(T)$, $f_p(v_j, v_i) \geqq s(T_i^{ij}) = f_q(v_i, v_j)$ and the conclusion follows. ■

Consider now the problem where finite capacities are associated with the edges of the tree. Let $\kappa(v_i, v_j) \in R_+$ be the capacity associated with edge (v_i, v_j) in the direction $v_i - v_j$. Formally, constraints

$$f_p(v_i, v_j) \leq \kappa(v_i, v_j) \text{ and } f_p(v_j, v_i) \leq \kappa(v_j, v_i), \forall (v_i, v_j) \in E, \tag{7}$$

are added to the optimization problem given by (1)-(3).

Suppose that each edge has the same capacity in both directions, i.e. $\kappa(v_i, v_j) = \kappa(v_j, v_i), \forall (v_i, v_j) \in E$. If we denote by f_p the unique flow pattern consisting of $f_p(v_i, v_j), f_p(v_j, v_i), \forall (v_i, v_j) \in E$, then lemma 2 implies

Theorem 3 : Let v_q be a minimax flow center for the uncapacitated problem. If f_q is feasible with respect to (7), then v_q is also a minimax flow center for the capacitated problem. If f_q is not feasible, then the capacitated problem has no feasible solution.

Let now $\kappa(v_i, v_j)$ and $\kappa(v_j, v_i)$ be unrelated and assume that v_q is the minimax flow center of the uncapacitated problem with the corresponding flow pattern f_q. If f_q is feasible with respect to all given capacities, then v_q is again optimal. Assume now that f_q is not feasible. Let $\overline{E}$ be the set of edges (v_i, v_j) with $f_q(v_i, v_j) > \kappa(v_i, v_j)$. For every edge in $\overline{E}$, define subtrees T_i^{ij} and T_j^{ij} as in the proof of lemma 2. Since each such infeasible edge implies that feasible minimax flow centers must be located in T_i^{ij}, we can state

Lemma 4 : If $\bigcap_{(v_i,v_j)\in\overline{E}} T_1^{ij} = \varnothing$, then the capacitated problem (1)-(3), (7) has no solution.

An immediate consequence of lemma 4 is that all infeasible edges must be located on a single path including v_q as an end point, if a feasible solution is to exist. Assume that infeasible edges are numbered as $(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), ..., (v_{i_r}, v_{j_r})$, so that there is a path $P_{qi_r} = (v_q, ..., v_{j_1}, v_{i_1}, ..., ..., v_{j_2}, v_{i_2}, ..., v_{j_r}, v_{i_r})$. Then the argument applied in the proof of lemma 4 together with the fact that flows accumulate towards the minimax flow center implies

Theorem 5 : Let P_{qi_r} be a path from the uncapacitated minimax flow center v_q to the end point of an infeasible edge v_{i_r}, so that P_{qi_r} includes all infeasible edges. Then either f_{i_r} is feasible in which case v_{i_r} is the optimal capacitated minimax center or f_{i_r} is infeasible and no feasible solution to (1)-(3), (7) exists.

Note that theorems 3 and 5 also allow sensitivity analyses concerning changes of edge capacities and supplies.

3. The Optimal Location of a Second Facility

In this section we will discuss the problem of locating a second facility assuming that the first facility is and remains located at the MMFC. Thus, the problem is one of conditional location. The objective is now to route the supplies to either of the two facilities, so as to minimize the largest edge flow. As before, let v_q be the MMFC and suppose that the second facility is located at v_p. Then one can prove

Lemma 6 : $v_p \in T_1^q$.

The proof is easy as any location at $v_p \notin T_1^q$ implies that the edge connecting T_1^q to v_q must carry $s(T_1^q)$ flow units as before, hence no improvement has been made.

With $v_p \in T_1^q$, the flows in subtrees $T_k^q, k \neq 1$, remain unchanged and the largest edge flow can occur one of the broken lines shown in Figure 1.

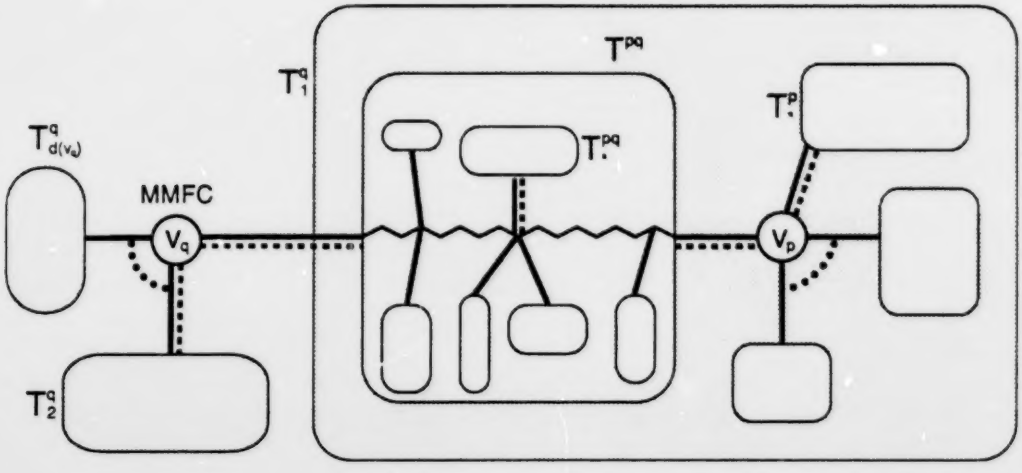

Figure 1

Note that there might exist several vertices in T_1^q such that locating the second facility at any of them would result in edge flows no larger than $s(T_2^q)$, the flow on the edge connecting T_2^q to v_q. In such a case, all these vertices would constitute optimal locations for the second facility. However, for the purposes of this section, we will determine a vertex v_p in T_1^q which minimizes the largest edge flow in the

subtree obtained by adding to T_1^q the vertex v_q and its connecting edge. Such a location will be optimal irrespective of the value of $s(T_2^q)$.

In T_1^q, the flows may be routed as follows. All units in T^{pq} are first shipped to the nearest point on the unique $(v_q - v_p)$ path. Half of the $s(T^{pq})$ units now available on that path are shipped to v_q and the other half to v_p. Define now subsubtrees $T_\ell^{pq} \subset T^{pq}$ as trees of the forest that is generated by deleting all edges and vertices of the $(v_q - v_p)$ path. The largest edge flow implied by the shipments then either occurs on the feeder edge that connects the largest subsubtree $T_\bullet^{pq}$ to the $(v_q - v_p)$ path or in the edges adjacent to v_q and v_p, respectively.

In $T_1^q \setminus T^{pq}$, all units are sent on their unique paths to v_p with the largest edge flow occuring on the edge that connects $T_\bullet^p$ to v_p where $T_\bullet^p$ is the second largest subtree generated by v_p (the largest one includes $\left(\bigcup_{k=2}^{d(v_q)} T_k^q \right) \cup \{v_q\} \cup (T^{pq})$). Then the critical edge flows in T_1^q are $z_1 = 1/2\ s(T^{pq}), z_2 = s(T_\bullet^{pq}), z_3 = s(T_\bullet^p)$ and we try to minimize $z = \max\{z_1, z_2, z_3\}$.

In order to do so, the following procedure is applied. Initially, locate v_p in T_1^q adjacent to v_q, so that $T^{pq} = \varnothing$ and $z = z_3$. Note that moving v_p along some path away from v_q increases z_1 and z_2 and decreases z_3. We will move v_p away from v_q one step at a time, always locating v_p in its second largest subtree $T_\bullet^p$ as otherwise there can be no improvement since the flow that determined z_3 in the previous step will now determine z_2. Suppose now that the facility to be located has been tentatively moved from v_p to $v_{p'}$, and let z_1', z_2', z_3' and z' be defined for $v_{p'}$ similarly to z_1, z_2, z_3 and z. The resulting subtrees are shown in Figure 2 where the solid (broken) lines

indicate the subtrees for $v_p(v_{p'})$.

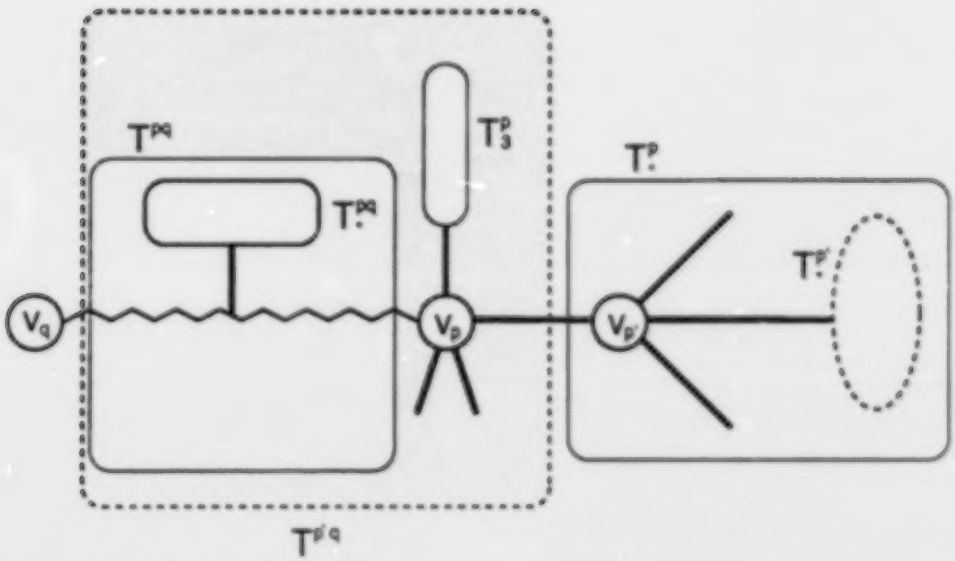

Figure 2

In this situation, $T^{p'q}_{\bullet}$ either equal $T^{pq}_{\bullet}$ or T^{p}_{3}, the latter being the third largest subtree emanating from v_p. Note that this implies that $z'_2 = z_2$ or $z'_2 \leq z_3$. It follows that $z'_2 \leq z$ in all situations. To derive a stopping criterion for our procedure, we may thus restrict our attention to z_1, z_3, z'_1 and z'_3.

Recall that for any tentative move of the facility away from v_q one has $z'_1 \geq z_1$ and $z'_3 \leq z_3$ and that initially $z = z_3$. The procedure should therefore terminate whenever $z'_1 > z_3$, i.e.

$$1/2\, s(T^{p'q}) > s(T^{p}_{\bullet}). \tag{8}$$

By construction, one has

$$s(T^{p'q}) + s(T_*^p) = s(T_1^q), \tag{9}$$

which implies that the desired condition for the optimal location of the second facility is

$$s(T_*^p) < 1/3\, s(T_1^q) \tag{10}$$

We now rapidly describe an efficient procedure for determining the optimal location of a second facility in T. This procedure is based on principles similar to those used in the derivation of Goldman's (1971) algorithm for finding medians in trees and of Eiselt, Gendreau and Laporte's (1991) method for locating facilities in networks with single edge failures.

Every pendent vertex v_i of T_1^q (except for v_q itself) is first "folded" on its unique neighbour (or "father") $\phi(v_i)$ by adding its weight $s(v_i)$ to $s(\phi(v_i))$ and deleting it from the graph. These deletions create "new" pendent vertices which can themselves be folded on their fathers. This process is repeated until only v_q remains in T_1^q. As we go along, we record for each vertex v_i the weight of the largest vertex folded on it and the name of this vertex $\rho(v_i)$. By construction, when a vertex v_j is folded on vertex v_i, its current weight $s(v_j)$ is equal to the weight of the subtree generated by v_i and rooted at v_j. Therefore, $s(\rho(v_i))$ corresponds to $s(T_*^i)$ in the previous description. The final step of the procedure consists in following the path $v_q, \rho(v_q), \rho(\rho(v_q)),\ldots$ until a vertex v_p satisfying condition (11) is reached.

The above algorithm can be implemented in $O(|\ T_1^2\ |)$ operations, if proper data structures, such as doubly-linked lists for representing sets of neighbours, are used.

4. The Optimal Addition of an Edge

The problem discussed in this section examines the effects of the addition of an edge to the given tree. Keeping the facility located at the MMFC determined prior to the introduction of the edge, some flows can be rerouted so as to decrease the largest edge flow. If it were possible to not only introduce a new edge but also change the location of the facility, then it may be possible to further decrease the edge flow as shown in Figure 3.

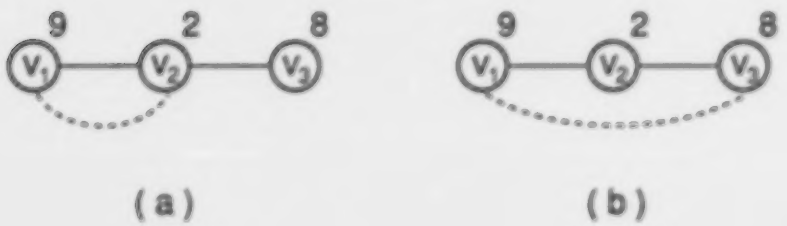

Figure 3

The numbers next to the vertices denote their respective supplies. In the given tree (without the broken edges) v_2 is the unique MMFC. Keeping the facility location at v_2, it is optimal to introduce a new edge (v_1, v_2) as shown in Figure 3a. The resulting maximal edge flow is $z^* = 8$. On the other hand, if the location of the facility can

also be varied, one may introduce the edge (v_1, v_3) and relocate the minimax flow center at v_1, resulting in a smaller edge flow of 5.

Define z^{*T} as the objective function value of the problem (1)-(3) and let $z^{**}\left(z^{*f}\right)$ denote the optimal objective value if an edge is added and the location of the facility is variable (fixed). Then we can state

Lemma 7 : $z^{*T} \geqq z^{*f} \geqq z^{**}$.

The proof of the first part of the lemma uses the fact that the addition of a new edge cannot increase the objective function value. The second part is also obvious since each solution to the fixed MMFC problem is also feasible for the variable MMFC problem.

Consider now the problem where the MMFC has been determined at v_q and one has to optimally locate a new edge. Simple inspection reveals that no additional edge (v_i, v_j) with $v_i, v_j \in T_k^q, k = 1, ..., d(v_q)$ or $v_i \in T_k^q, v_j \in T_\ell^q$ with $k \neq 1 \neq \ell$ will decrease $z*$. In other words, exactly one of the vertices of the new edge must be in the subtree T_1^q. Select any $v_1^* \in T_1^q$, then the new edge is either (v_1^*, v_q) or (v_1^*, v_k^*) with $v_k^* \in T_k^q$ for some $k \neq 1$. These two possibilities are displayed in Figure 4 as the dotted and the broken edge, respectively, where $v_q' \in T_1^q$ and $v_q'' \in T_k^q$ are vertices adjacent to v_q. In either case, the new graph is unicyclic. The cycle generated by the broken edge is $(v_q, v_q', ..., v_1^*, v_k^*, ..., v_q'', v_q)$ and the cycle generated by the dotted edge is $(v_q, v_q', ..., v_1^*, v_q)$. The bold edges in T_1^q and T_k^q are the feeder edges that connect the subsubtrees in T_1^q and T_k^q to the respective cycle. Similar to the previous section, the subsubtrees in T_1^q are defined as the forest that results from T_1^q if all edges on the

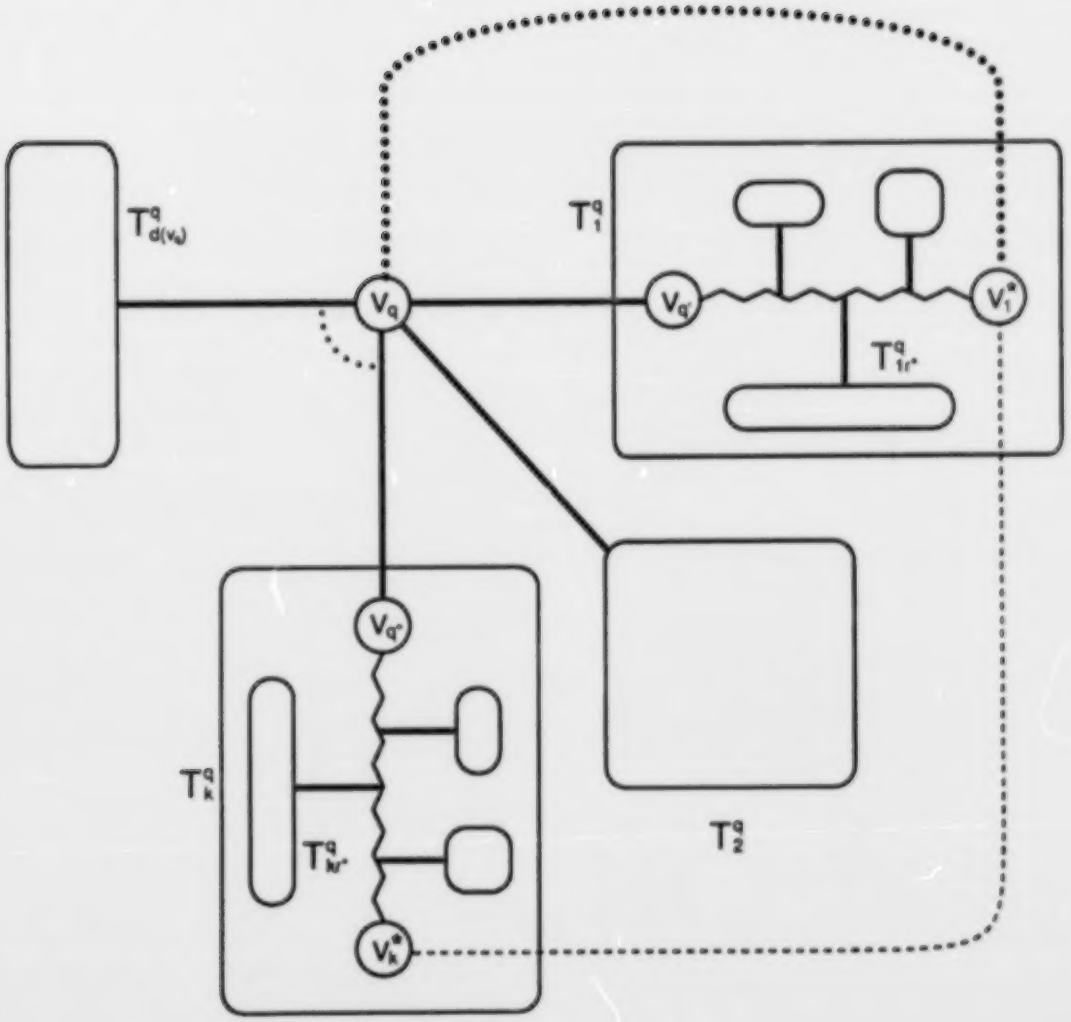

Figure 4

$(v'_q - v^*_1)$ path are removed, and similarly for T^q_k. Let $T^q_{1r*} \subset T^q_1$ and $T^q_{kr*} \subset T^q_k$ be the respective largest subsubtrees. The optimal flow pattern in the two respective cases is easily determined by shipping all supplies in the subsubtrees to their nearest vertex on the cycle and then $1/2\left[s(T^q_1) + s(T^q_k)\right]$ units in case of the broken edge and $1/2\ s(T^q_1)$ units in case of the dotted edge are shipped clockwise and the same amounts counterclockwise to the minimax flow center v_q (cancelling out flows in opposite directions on any given edge whenever they occur). The flows within the other subtrees remain unchanged. As far as the largest edge flow is concerned one has to distinguish between the two cases. For notational convenience, let (T^q_l, v_q) denote the unique edge connecting the subtree T^q_l with the vertex v_q before the addition of the new edge.

Case 1 : Edge (v_q, v^*_1) is introduced. Then the largest edge flow occurs either

(a) in the edges (v'_q, v_q) and (v^*_1, v_q) with flow

$$z^*_1 = f^*_q(v'_q, v_q) = f^*_q(v^*_1, v_q) = 1/2\ s(T^q_1)$$

or (b) in the edge (T^q_2, v_q) with flow $z^*_2 = f^*_q(T^q_2, v_q) = s(T^q_2)$

or (c) in one of the feeder edges which, due to the order introduced above, is the one connecting T^q_{1r*} with the $(v'_q - v^*_1)$ path; its flow is $z^*_3 = s(T^q_{1r*})$.

Case 2 : Edge (v^*_1, v^*_k) is introduced. Then the largest edge flow occurs either

(a) in the edges (v_q, v'_q) and (v_q, v''_q) with flows

$$z^{**}_1 = f^{**}_q(v'_q, v_q) = f^{**}_q(v''_q, v_q) = 1/2\left[s(T^q_1) + s(T^q_k)\right]$$

or (b) in the edge (T^q_2, v_q) with flow $z^{**}_2 = s(T^q_2)$, (if $k = 2$, i.e. if the new

edge connects T_1^q and T_2^q, then the maximal flow may occur in the edge (T_3^q, v_q) with flow $z_2^{**} = s(T_3^q)$)

or (c) in any of the feeder edges in T_1^q or T_k^q with the largest flow being either $s(T_{1r*}^q)$ or $s(T_{kr*}^q)$, i.e. $z_3^{**} = \max\{s(T_{1r*}^q); s(T_{kr*}^q)\}$.

Thus the largest edge flows in the two respective cases are $z^* = \max\{z_1^*, z_2^*, z_3^*\}$ and $z^{**} = \max\{z_1^{**}, z_2^{**}, z_3^{**}\}$. Then one can prove

Lemma 8 : $z^* \leq z^{**}$.

Proof : Suppose the lemma is not true and $z^* > z^{**}$. Since $z_1^* \leq z_1^{**}, z_2^* \geq z_2^{**}$ and $z_3^* \leq z_3^{**}$, we must have $z^* = z_2^*$, otherwise the assumption cannot be satisfied. In order to satisfy the strict inequality, $z_2^* > z_1^{**}, z_2^* > z_2^{**}$, and $z_2^* > z_3^{**}$. The second of these conditions is only satisfied if $k = 2$. Using this result in $z_2^* > z_1^{**}$ implies that $s(T_2^q) > 1/2[s(T_1^q) + s(T_2^q)]$ or $s(T_2^q) > s(T_1^q)$ which contradicts the order of the subtrees; thus the lemma. ■

Since $z^{*I} = \min\{z^*; z^{**}\}$, lemma 8 implies immediately

Corollary 9 : For a given minimax flow center at v_q and any given vertex $v_1^* \in T_1^q$ it is never worse to connect v_1^* with v_q than with any other vertex in $T \setminus T_1^q$ in order to minimize the maximal edge flow.

Let now v_1^* be the MMFC of subtree T_1^q considered in isolation. We construct a flow pattern $\overline{f}_q$ in the graph that results from the original tree T after the edge (v_q, v_1^*) has been added to it. We then show that the addition of the edge (v_q, v_1^*) is optimal. First, all supplies in T_1^q are shipped to v_1^* on their respective unique paths. In the resulting flow pattern no edge can have a flow in excess of $1/2\, s(T_1^q)$ due to theorem

1, applied to subtree T_1^q. Now $s(T_1^q)$ units are available at v_1^*. Half of this amount is sent along the new edge to v_q, i.e. $\overline{f}_q(v_1^*, v_q) = 1/2\ s(T_1^q)$. The other half is sent to v_q on the unique $(v_q^* - v_q)$ path being in part cancelled out by flows which were sent along the same path but in opposite direction when shipped to v_1^*. Thus no edge on the $(v_1^* - v_q)$ path and, more generally, in or adjacent to T_1^q carries a flow of more than $1/2\ s(T_1^q)$. In particular, $\overline{f}_q(v', v_q) = \overline{f}_q(v_1^*, v_q) = 1/2\ s(T_1^q)$. Thus the largest edge flow in the graph is $\max\{1/2\ s(T_1^q); s(T_2^q)\} = \max\{z_1^*, z_2^*\} \leq \max\{z_1^*, z_2^*, z_3^*\} = z^* = z^{*f}$ and hence the new edge must be located optimally. This implies

Theorem 10 : Let v_q be the minimax flow center in T and let v_1^* be the minimax flow center in T_1^q. For a fixed MMFC at v_q an edge is optimally introduced in the sense that it minimizes the largest flow in any edge if it connects v_q and v_1^*. The largest edge flow is then

$$z^{*f} \quad = \quad \max\{1/2\ s(T_1^q); s(T_2^q)\}.$$

As an MMFC can be determined in linear time (see section 2), the problem of optimally adding an edge can also be solved in $O(n)$ time.

Some of the above results can also be used to address a different problem. Suppose that a new edge (v_i, v_j) has been arbitrarily introduced in the tree, resulting in a unicyclic graph U. One problem is now to optimally relocate the facility. Clearly, this problem is equivalent to finding a minimax flow center in a unicyclic graph but knowledge of the MMFC prior to introducing the new edge proves quite helpful in the process. Denote by v_q the MMFC in T and by v_p the MMFC in U. Furthermore, let C be the set of vertices on the unique cycle in U. Then it was already shown

that the largest edge flow towards v_q cannot decrease if not exactly one of v_i and v_j is in T_1^q. This finding can be strengthened to

Lemma 11 : If $v_i, v_j \in T_1^q$ or $v_i, v_j \notin T_1^q$, then $v_p = v_q$.

Proof : If $v_p \in T_k^q, k \neq 1$, then the resulting flow pattern is such that $f_p(v_{q'}, v_q) = s(T_1^q)$. If $v_p \in T_1^q$, then $f_p(v_q, v_{q'}) = s(T) - s(T_1^q) \geq s(T_1^q)$ (since $s(T_1^q) \leq 1/2 s(T)$ by theorem 1). In either case, no improvement is gained by relocating the facility. ■

A similar argument can be used to prove

Lemma 12 : If $v_i \in T_1^q$ and $v_j \notin T_1^q$, then $v_p \in C$.

In this case, v_p may or may not coincide with v_q. Consequently, given v_q one could find v_p in U by first examining the condition of lemma 11. If it is satisfied, then $v_p = v_q$; otherwise v_p is determined among the vertices of C.

5. Conclusion

In this paper we have addressed the issue of sensitivity analysis of minimax flow centers in trees. In most cases, closed form characterizations of the conditions could be described. Even where an algorithm may be required to establish, say, the MMFC in a unicyclic graph, one may expect this to be done rather efficiently. The reason for this are the encouraging computational results reported by Eiselt (1990) concerning 2–MMFC problems. Future research could investigate the location and sensitivity of MMFC's in other graphs.

6. Acknowledgments

This research was supported by grants OGP0009160 and OGP0038816 from the Natural Sciences and Engineering Research Council of Canada. This support is gratefully acknowledged. The authors would like to thank Professor Gilbert Laporte for many helpful discussions and René Séguin for his valuable comments on an earlier version of this paper. Finally, we wish to express our gratitude to the members of the Department of General Business Management and Personnel Management of the Chinese University of Hong Kong for their support during the first author's sabbatical stay.

7. References

Ahuja, R.K. (1986), "Algorithms for the Minimax Transportation Problem", *Naval Research Logistics Quarterly* **33**, 725–739.

Eiselt, H.A. (1985), "Location Problems in Trees with Minimax Objective, Parts I and II", *Proceedings of the ASAC*, vol. 6, part 2, 1–29.

Eiselt, H.A. (1990), "Computational Experience with Minimax Flow Centers in Trees", *Computing* **43**, 297–307.

Eiselt, H.A., Gendreau, M. (1990), "On the Determination of Minimax Flow Centers in Trees", publication #738, Centre de recherche sur les transports, Université de Montréal.

Eiselt, H.A., Gendreau, M. (1991), "An Optimal Algorithm for Weighted Minimax Flow Centers on Trees", to appear in *Transportation Science*.

Eiselt, H.A., Gendreau, M., Laporte, G. (1991), "Location of Facilities on Networks Subject to Single Edge Failures", to appear in *Networks*.

Eiselt, H.A., Pederzoli, G. (1984), "A Location Problem in Graphs", *New Zealand Operational Research* **12**, 49–53.

Finke, G., Röck, H. (1983), "A Primal Algorithm for a Class of Minimax Network Flow Problems", *Methods of Operations Research* **45**, 221–232.

Fujishige, A., Nakayama, A., Cui, W.-T. (1986), "On the Equivalence of the Maximum Balanced Flow Problem and the Weighted Minimax Flow Problem", *Operations Research Letters* **5**, 207–209.

Goldman, A.J. (1971), "Optimal Center Location in Simple Networks", *Transportation Science* **5**, 212–221.

Gondran, M., Minoux, M. (1984), *Graphs and Algorithms*, J. Wiley & Sons, New York.

Hakimi, S.L. (1964), "Optimal Locations of Switching Centers and the Absolute Centers and Medians of a Graph", *Operations Research* **12**, 450–459.

Handler, G.Y., Mirchandani, P.B. (1979), *Locations on Networks: Theory and Algorithms*, The MIT Press, Cambridge, Mass.

Ichimori, T., Ishii, I., Nishida, T. (1981), "Weighted Minimax Real-Valued Flows", *Journal of the Operations Research Society of Japan* **24**, 52–60.

Minoux, M. (1976), "Flots équilibrés et flots avec sécurité", *Bulletin de la Direction des Études et Recherches, Série – Mathématiques, Informatique* **1**, 5–16.

Tamir, A. (1982), "On the Solution of Discrete Bottleneck Problems", *Discrete Applied Mathematics* **4**, 299–302.